Tutorial 4: Selected problems of Assignment 4, 5

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Announcements

(1) HW1 - HW4 are marked and are ready to pick up.

(2) Extra office hour for Midterm: 18 Oct (Thurs) 14:00 - 17:00

Hölder's and Minkowski's inequalities

(I) Euclidean space version: $X = \mathbb{R}^n$; $\forall p \geq 1$, define p-norm

$\|\cdot\|_p : X \to \mathbb{R}$ by $a = (a_1, \cdots, a_n) \mapsto \|a\|_p := \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}}$

(a) Hölder's inequality: $\forall p, q > 1$ w/ $\frac{1}{p} + \frac{1}{q} = 1$, $\forall a, b \in X$,

$$\|\underbrace{a \cdot b}_{= (a_1 b_1, \cdots, a_n b_n)}\|_1 \leq \|a\|_p \|b\|_q$$

(b) Minkowski's inequality: $\forall p \geq 1$, $\forall a, b \in X$,

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p$$

(II) Function space version: $X = R[a,b]$ (say real-valued), $\forall p \geq 1$,

define "p-norm" $\|\cdot\|_p : X \to \mathbb{R}$ by

$$f \mapsto \|f\|_p := \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}}, \quad f \in X$$

(a) Hölder's inequality: $\forall p, q > 1$ w/ $\frac{1}{p} + \frac{1}{q} = 1$, $\forall f, g \in X$,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

(b) Minkowski's inequality: $\forall p \geq 1$, $\forall f, g \in X$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Rmk: Both versions are special cases of inequalities for "measure spaces".

Q1) (Ex.4, Q8)

(a) $\forall\, 1 \leq p < +\infty$, define $\ell^p := \{ (a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R} ; \sum_{n=1}^{\infty} |a_n|^p < +\infty \}$

and define p-norm $\|\cdot\|_p : \ell^p \to \mathbb{R}$ by

$$\begin{array}{ccc} \ell^p & \to & \mathbb{R} \\ \cup & & \\ a = (a_n) & \mapsto & \|a\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \end{array}$$

Show that $(\ell^p, \|\cdot\|_p)$ is a normed space.

(b) $p = +\infty$: define $\ell^{\infty} := \{ (a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R} ; \sup_n |a_n| < +\infty \}$

and define sup-norm $\|\cdot\|_{\infty} : \ell \to \mathbb{R}$ by

$$\begin{array}{ccc} \ell & \to & \mathbb{R} \\ \cup & & \\ a = (a_n) & \mapsto & \|a\|_{\infty} := \sup_n |a_n| \end{array}$$

Show that $(\ell^{\infty}, \|\cdot\|_{\infty})$ is a normed space.

Sol'n: (a) Check the axioms [N1]-[N3] for normed spaces:

[N1]: $\forall\, a \in \ell^p$, $\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \geq 0$, which

$(=0)$ holds $\Leftrightarrow \forall n,\ |a_n| = 0 \Leftrightarrow a = 0$

[N2]: $\forall\, a \in \ell^{\infty}$, $\forall \alpha \in \mathbb{R}$, $\|\alpha a\|_p = \left(\sum_{n=1}^{\infty} |\alpha a_n|^p \right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |\alpha|^p |a_n|^p \right)^{\frac{1}{p}}$

$= |\alpha| \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} = |\alpha| \|a\|_p$

[N3] : $\forall a, b \in \ell^p$, $\forall N \in \mathbb{N}$, write $a^{(N)} = (a_1, \cdots, a_N)$; $b^{(N)} = (b_1, \cdots, b_N)$

then $\left(\sum_{n=1}^{N} |a_n + b_n|^p\right)^{\frac{1}{p}} = \|a^{(N)} + b^{(N)}\|_p \leq \|a^{(N)}\|_p + \|b^{(N)}\|_p$ (by (Ib))

$= \left(\sum_{n=1}^{N} |a_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |b_n|^p\right)^{\frac{1}{p}} \leq \|a\|_p + \|b\|_p$

$\therefore$ Take $N \to +\infty$: $\|a+b\|_p \leq \|a\|_p + \|b\|_p$

$\therefore$ $(\ell^p, \|\cdot\|_p)$ is a normed space.

(b) [N1] : $\forall a \in \ell^\infty$, $\|a\|_\infty = \sup_n |a_n| \geq 0$, which

$(=0)$ holds $\Leftrightarrow \forall n, |a_n| = 0 \Leftrightarrow a = 0$

[N2] : $\forall a \in \ell^\infty$, $\forall \alpha \in \mathbb{R}$, $\|\alpha a\|_\infty = \sup_n |\alpha a_n| = |\alpha| \sup_n |a_n| = |\alpha| \|a\|_\infty$

[N3] : $\forall a, b \in \ell^\infty$, $\forall n \in \mathbb{N}$, $|a_n + b_n| \leq |a_n| + |b_n| \leq \sup_m |a_m| + \sup_m |b_m| = \|a\|_\infty + \|b\|_\infty$

$\therefore$ $\|a+b\|_\infty = \sup_n |a_n + b_n| \leq \|a\|_\infty + \|b\|_\infty$

$\therefore$ $(\ell^\infty, \|\cdot\|_\infty)$ is a normed space.

Q2) (Ex. 4, Q6)

Under same notations as in Q1, show that for $0<p<1$,

$(\ell^p, \|\cdot\|_p)$ is NOT a normed space.

Sol: Showing [N3] is false: Choose $a=(1,0,\cdots)$; $b=(0,1,0,\ldots)$,

then $\|a\|_p = 1 = \|b\|_p$; $\|a+b\|_p = (1^p+1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$

$\because$ $0<p<1 \Rightarrow \|a+b\|_p = 2^{\frac{1}{p}} > 2 = \|a\|_p + \|b\|_p$

$\therefore$ [N3] is false, hence $(\ell^p, \|\cdot\|_p)$ is NOT a normed space.

Rmk: Exactly the same argument shows that $\forall n\geq 2$, $\forall 0<p<1$,

$(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a normed space.

Q3) (Ex. 5, Q2)

Let $X = C[a,b]$ be the space of continuous functions on $[a,b]$.

$\forall p \geq 1$, $\|\cdot\|_p : C[a,b] \subseteq R[a,b] \longrightarrow \mathbb{R}$ defined as in (II)

(Exercise: $(X, \|\cdot\|_p)$ is a normed space)

and $d_p : X \times X \to \mathbb{R}$ be the induced metric.

Show that $\forall p > 1$, d_p is stronger but inequivalent to d_1.

Sol: (1) d_p is stronger than d_1: $\forall f, g \in X$,

$$d_1(f,g) = \|f-g\|_1 = \|(f-g)\cdot 1\|_1 \leq \|f-g\|_p \cdot \|1\|_q \quad \text{(by IIa)}$$

$$= C d_p(f,g), \text{ where } C = \|1\|_q = (b-a)^{\frac{1}{q}}.$$

(2) d_p is inequivalent to d_1: Suppose on the contrary they are equivalent, then

$\exists c, C \in \mathbb{R}$ s.t. $\forall (f_n) \subseteq X$, $\forall n$, $c\, d_p(f_n, 0) \leq d_1(f_n, 0) \leq C d_p(f_n, 0)$

i.e. $c\|f_n\|_p \leq \|f_n\|_1 \leq C\|f_n\|_p$.

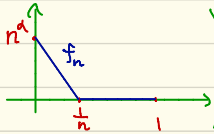
However, we will construct (f_n) s.t. $\|f_n\|_p \to \infty$ and $\|f_n\|_1 \to 0$

which is a contradiction, so d_p is inequivalent to d_1.

Constructing (f_n): (for simplicity assume $[a,b]=[0,1]$)

Fix $\frac{1}{p}<\alpha<1$, define $f_n:[0,1]\to\mathbb{R}$ by

$$f_n(x)=\begin{cases}-n^{\alpha+1}x+n^{\alpha}, & x\in[0,\frac{1}{n}]\\ 0, & x\in[\frac{1}{n},1]\end{cases}$$

(Picture: 

)

then $\|f_n\|_1 = \int_0^{\frac{1}{n}}(-n^{\alpha+1}x+n^{\alpha})\,dx = n^{\alpha}\int_0^{\frac{1}{n}}(1-nx)\,dx = n^{\alpha}\left[-\frac{(1-nx)^2}{2n}\right]_0^{\frac{1}{n}}$

$= n^{\alpha}\cdot\left(\frac{1}{2n}\right) = \frac{1}{2}n^{\alpha-1}\to 0$ as $n\to\infty$ ($\because \alpha<1$)

but $(\|f_n\|_p)^p = \int_0^{\frac{1}{n}}(-n^{\alpha+1}x+n^{\alpha})^p\,dx = n^{\alpha p}\left[-\frac{(1-nx)^{p+1}}{(p+1)n}\right]_0^{\frac{1}{n}}$

$= n^{\alpha p}\cdot\left(\frac{1}{(p+1)n}\right) = \frac{1}{p+1}n^{\alpha p-1}\to+\infty$ as $n\to+\infty$ ($\because \alpha>\frac{1}{p}$)

$\therefore \|f_n\|_p\to\infty$ as $n\to\infty$

Hence (f_n) is the desired counterexample.